Linear Algebra

1. (10%) (∗∗)
   For which value(s) of a does the following system have zero solutions? One solution? Infinitely many solutions?

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_3 &= 2 \\ (a^2 - 4)x_3 &= a - 2 \end{aligned}$$

2. (10%) (∗)
   How should the coefficients a, b. and c be chosen so that the system

$$\begin{aligned} ax + by - 3z &= -3 \\ -2x - by + cz &= -1 \\ ax + 3y - cz &= -3 \end{aligned}$$

   has the solution $x = 1$, $y = -1$, and $z = 2$?

3. (10%) (∗ ∗ ∗)
   Prove: If $\mathbf{B}$ is invertible, then $\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A}$ if and only if $\mathbf{AB} = \mathbf{BA}$.

4. (10%) (∗ ∗ ∗)
   Prove: If $\mathbf{A}$ is invertible, then $\mathbf{A}+\mathbf{B}$ and $\mathbf{I}+\mathbf{BA}^{-1}$are both invertible or both not invertible.

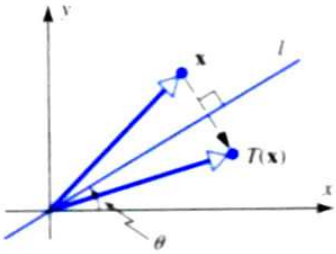
5. (10%) (∗∗)
   Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose that $\mathbf{B}_1$ is obtained by adding the same number t to each entry in the ith row of $\mathbf{A}$ and that $\mathbf{B}_2$ is obtained by subtracting t from each entry in the ith row of $\mathbf{A}$. Show that

$$\det(\mathbf{A}) = \frac{1}{2}[\det(\mathbf{B}_1) + \det(\mathbf{B}_2)].$$

6. (10%) (∗)
   Find the distance between the given parallel planes.
   (a) (3.3%) $3x - 4y + z = 1$ and $6x - 8y + 2z = 3$
   (b) (3.3%) $-4x + y - 3z = 0$ and $8x - 2y + 6z = 0$
   (c) (3.3%) $2x - y + z = 1$ and $2x - y + z = -1$

7. (10%) (∗)
   Find the standard matrix for the stated composition of linear operators on R^3.
   (a) (3.3%) A reflection about the yz-plane, followed by an orthogonal projection on the xz-plane.
   (b) (3.3%) A rotation of 45^o about the y-axis, followed by a dilation with factor $k = \sqrt{2}$.
   (c) (3.3%) An orthogonal projection on the xy-plane, followed by a reflection about the yz-plane.

8. (10%) (∗∗)
Let l be the line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \leq \theta < \pi$. Let T: $R^2 \rightarrow R^2$ be the linear operator that reflects each vector about l (see the accompanying figure)



(a) (5%) Find the standard matrix for T

(b) (5%) Find the reflection of the vector $x = (1, 5)$ about the line l through the origin that makes an angle of $\theta = 30^o$ with the positive x-axis

9. (10%) ($* * *$)
Show that the set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with addition defined by $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and scalar multiplication defined by $k \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$ is a vector space. What is the zero vector in this space?

10. (10%) (∗)
Show that the vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$,and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in $\mathbf{R}^4$.

11. (10%) (∗)
Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space $\mathbf{V}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

12. (10%) (∗∗)
Find a basis for the subspace of $\mathbf{P}_2$ spanned by the given vectors.
$1 + x - 3x^2$, $2 + 2x - 6x^2$, $3 + 3x - 9x^2$

13. (10%) (∗∗)
Find a subset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

(a) (3.3%) $\mathbf{v}_1 = (1, 0, 1, 1)$, $\mathbf{v}_2 = (-3, 3, 7, 1)$, $\mathbf{v}_3 = (-1, 3, 9, 3)$, $\mathbf{v}_4 = (-5, 3, 5, -1)$

(b) (3.3%) $\mathbf{v}_1 = (1, -2, 0, 3)$, $\mathbf{v}_2 = (2, -4, 0, 6)$, $\mathbf{v}_3 = (-1, 1, 2, 0)$, $\mathbf{v}_4 = (0, -1, 2, 3)$

(c) (3.3%) $\mathbf{v}_1 = (1, -1, 5, 2)$, $\mathbf{v}_2 = (-2, 3, 1, 0)$, $\mathbf{v}_3 = (4, -5, 9, 4)$, $\mathbf{v}_4 = (0, 4, 2, -3)$,$\mathbf{v}_5 = (-7, 18, 2, -8)$

14. (10%) (∗∗)
What conditions must be satisfied by b_1, b_2, b_3, b_4. and b_5 for the overdetermined linear system

$$\begin{aligned} x_1 - 3x_2 &= b_1 \\ x_1 - 2x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 - 4x_2 &= b_4 \\ x_1 + 5x_2 &= b_5 \end{aligned}$$

to be consistent?

15. (10%) (∗∗)
For what values of s is the solution space of

$$\begin{aligned} x_1 + x_2 + sx_3 &= 0 \\ x_1 + sx_2 + x_3 &= 0 \\ sx_1 + x_2 + x_3 &= 0 \end{aligned}$$

the origin only, a line through the origin, a plane through the origin, or all of $\mathbf{R}^3$?

16. (10%) (∗)
Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

(a) (5%) Find bases for the row space and nullspace of $\mathbf{A}$.

(b) (5%) Find bases for the column space of $\mathbf{A}$ and nullspace of $\mathbf{A}^T$

17. (10%) (∗ ∗ ∗)
Prove: If $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices and $\mathbf{A}$ is an $n \times n$ matrix, then

$$(\mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{u})^2 \leq (\mathbf{u}^T\mathbf{A}^T\mathbf{A}\mathbf{u})(\mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v})$$

18. (10%) (∗∗)
Let $\mathbf{R}^3$ have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by $(0, 1, 2)$,$(-1, 0, 1)$,$(-1, 1, 3)$.

19. (10%) (∗∗)
Let $\mathbf{W}$ be the plane with equation $5x - 3y + z = 0$.

(a) (2.5%) Find a basis for $\mathbf{W}$.

(b) (2.5%) Find the standard matrix for the orthogonal projection onto $\mathbf{W}$.

(c) (2.5%) Use the matrix obtained in (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ onto $\mathbf{W}$.

(d) (2.5%) Find the distance between the point $P_0(1, -2, 4)$ and the plane $\mathbf{W}$, and check your result.

20. (10%) $(**)$
Consider the bases $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ for P_1. where

$$\mathbf{p}_1 = 6 + 3x, \mathbf{p}_2 = 10 + 2x, \mathbf{q}_1 = 2, \mathbf{q}_2 = 3 + 2x$$

(a) (2.5%) Find the transition matrix from B' to B.

(b) (2.5%) Find the transition matrix from B to B'.

(c) (2.5%) Compute the coordinate vector $[\mathbf{p}]_B$. where $\mathbf{p} = -4 + x$, and compute $[\mathbf{p}]_{B'}$.

(d) (2.5%) Check your work by computing $[\mathbf{p}]_{B'}$ directly.

21. (10%) $(* * *)$
Let $\mathbf{V}$ be the space spanned by $\mathbf{f}_1 = \sin x$ and $\mathbf{f}_2 = \cos x$

(a) (2%) Show that $\mathbf{g}_1 = 2\sin x + \cos x$ and $\mathbf{g}_2 = 3\cos x$ form a basis for $\mathbf{V}$.

(b) (2%) Find the transition matrix from $B' = \{\mathbf{g}_1, \mathbf{g}_2\}$ to $B = \{\mathbf{f}_1, \mathbf{f}_2\}$.

(c) (2%) Find the transition matrix from B to B'.

(d) (2%) Compute the coordinate vector $[\mathbf{h}]_B$, where $\mathbf{h} = 2\sin x - 5\cos x$, and compute $[\mathbf{h}]_{B'}$

(e) (2%) Check your work by computing $[\mathbf{h}]_{B'}$ directly.

22. (10%) $(* * *)$
Find a, b, and c such that the matrix

$$A = \begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal. Are the values of a, b, and c unique? Explain.

23. (10%) $(**)$
Find a weighted Euclidean inner product on R^n such that the vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, 0, \ldots, 0) \\ \mathbf{v}_2 &= (0, \sqrt{2}, 0, \ldots, 0) \\ \mathbf{v}_3 &= (0, 0, \sqrt{3}, \ldots, 0) \\ &\vdots \\ \mathbf{v}_n &= (0, 0, 0, \ldots, \sqrt{n}) \end{aligned}$$

form an orthonormal set.

24. (10%) $(**)$

Find A^n if n is a positive integer and

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

25. (10%) $(**)$

Find a matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$, and determine $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$\mathbf{A} = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$$

26. (10%) $(**)$

Does there exist a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If so, find such a matrix; if not, explain why not

27. (10%) $(*)$

In advanced linear algebra, one proves the **Cayley-Hamilton Theorem**, which states that a square matrix $\mathbf{A}$ satisfies its characteristic equation; that is, if

$$\mathbf{c}_0 + \mathbf{c}_1\lambda + \mathbf{c}_2\lambda^2 + \cdots + \mathbf{c}_{n-1}\lambda^{n-1} + \lambda^n = 0$$

is the characteristic equation of A. then

$$\mathbf{c}_0\mathbf{I} + \mathbf{c}_1\mathbf{A} + \mathbf{c}_2\mathbf{A}^2 + \cdots + \mathbf{c}_{n-1}\mathbf{A}^{n-1} + \mathbf{A}^n = 0$$

Verify this result for

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

28. (10%) $(* * *)$

Find a 3×3 matrix $\mathbf{A}$ that has eigenvalues $\lambda = 0$, 1, and-1 with corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

respectively.

29. (10%) $(*)$
Determine whether the function is a linear transformation. Justify your answer.
$T : M_{22} \to R$

$$\mathbf{T}\left(\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}\right) = \mathbf{a}^2 + \mathbf{b}^2$$

30. (10%) $(**)$

(a) (5%) Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations. Define the functions $(T_1 + T_2) : V \to W$ and$(T_1 - T_2) : V \to W$ by

$$(\mathbf{T}_1 + \mathbf{T}_2)(\mathbf{v}) = \mathbf{T}_1(\mathbf{v}) + \mathbf{T}_2(\mathbf{v})$$
$$(\mathbf{T}_1 - \mathbf{T}_2)(\mathbf{v}) = \mathbf{T}_1(\mathbf{v}) - \mathbf{T}_2(\mathbf{v})$$

Show that $T_1 + T_2$ and $T_1 - T_2$ are linear transformations.

(b) (5%) Find $(T_1 + T_2)(x, y)$ and $(T_1 - T_2)(x, y)$ if $T_1 : \mathbf{R}^2 \to \mathbf{R}^2$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ are given by the formulas $T_1(x, y) = (2y, 3x)$ and $T_2(x, y) = (y, x)$.

31. (10%) $(**)$
Let $\mathbf{T}$ be multiplication by the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Find

(a) (2.5%) a basis for the range of $\mathbf{T}$

(b) (2.5%) a basis for the kernel of $\mathbf{T}$

(c) (2.5%) the rank and nullity of $\mathbf{T}$

(d) (2.5%) the rank and nullity of $\mathbf{A}$

32. (10%) $(**)$
Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be multiplication by

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$$

(a) (5%) Show that the kernel of T is a line through the origins, and find parametric equations for it.

(b) (5%) Show that the range of T is a plane through the origin, and find an equation for it.

33. (10%) $(***)$
Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be the matrix of $T : \mathbf{P}_2 \to \mathbf{P}_2$ with respect to the basis $\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = 3x + 3x^2, \mathbf{v}_2 = -1 + 3x + 2x^2, \mathbf{v}_3 = 3 + 7x + 2x^2$$

(a) (2.5%) Find $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $[T(\mathbf{v}_3)]_B$.

(b) (2.5%) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$.

(c) (2.5%) Find a formula for $T(a_0 + a_1 x + a_2 x^2)$.

(d) (2.5%) Use the formula obtained in (c) to compute $T(1 + x^2)$.

34. (10%) $(***)$
Let $T_1 : \mathbf{P}_1 \to \mathbf{P}_2$ be the linear transformation defined by

$$\mathbf{T}_1(\mathbf{p}(x)) = x\mathbf{p}(x)$$

and let $T_2 : \mathbf{P}_2 \to \mathbf{P}_2$ be the linear operator defined by

$$\mathbf{T}_2(\mathbf{p}(x)) = \mathbf{p}(2x + 1)$$

Let $B = \{1, x\}$ and $B' = \{1, x, x^2\}$ be the standard bases for P_1 and P_2.

(a) (3.3%) Find $[\mathbf{T}_2 \circ \mathbf{T}_1]_{B',B}$,$[\mathbf{T}_2]_{B'}$,and$[\mathbf{T}_1]_{B',B}$.

(b) (3.3%) State a formula relating the matrices in part (a).

(c) (3.3%) Verify that the matrices in part (a) satisfy the formula you stated in part (b).

35. (10%) $(**)$
Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for a vector space V. and let $T : V \to V$ be a linear operator such that

$$[\mathbf{T}]_B = \begin{bmatrix} -3 & 4 & 7 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

Find $[\mathbf{T}]_{B'}$, where $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the basis for V defined by

$$\mathbf{v}_1 = \mathbf{u}_1, \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_3 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

36. (10%) $(**)$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

(a) (3.3%) Find an LU-decomposition of $\mathbf{A}$.

(b) (3.3%) Express $\mathbf{A}$ in the from $\mathbf{A} = L_1 D U_1$, where L_1 is lower triangular with l's along the main diagonal, U_1 is upper triangular, and D is a diagonal matrix.

(c) (3.3%) Express $\mathbf{A}$ in the form $\mathbf{A} = L_2 U_2$, where L_2 is lower triangular with ls along the main diagonal and U_2 is upper triangular.

37. (10%) **Row-Echelon Form** ($*$)
Let A and I be $m \times m$ matrices. If $[\ I \mid M\ ]$ is the row-echelon form of $[\ A \mid I\ ]$, prove that A is nonsingular and $M = A^{-1}$.

38. (10%) **Matrix Inverse** ($*$)
Let A and B be invertible $m \times m$ matrices such that $A^{-1} + B^{-1}$ is also invertible, show that $A + B$ is also invertible. What is $(A+B)^{-1}$?

39. (10%) **Interpolations** ($*$)
Identify all the cubic polynomials of the form

$$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

for some real coefficients a_i such that the graph $y = f(x)$ passes through the points $(1, 0)$ and $(2, -15)$ in the xy-plane.

40. (10%) **Liner Transformation** ($**$)
Let the linear transformation $\mathbf{w} = T(\mathbf{v})$ from R^2 to R^3 be defined by

$$w_1 = v_1 - v_2, \quad w_2 = 2v_1 + v_2, \quad \text{and} \quad w_3 = v_1 - 2v_2$$

where $\mathbf{v} = [\ v_1 \quad v_2\ ]^T$ and $\mathbf{w} = [\ w_1 \quad w_2 \quad w_3\ ]^T$. Find the matrix A that represents T with respect to the ordered bases $B = \{\mathbf{e}_1; \mathbf{e}_2\}$ for R^2 and $C = \{\mathbf{e}_1; \mathbf{e}_2; \mathbf{e}_3\}$ for R^3. Check by computing $T([\ 2 \quad 1\ ]^T)$ two ways.

41. (10%) **Projection vs. Linear Transformation** ($**$)
Let $\mathbf{v}$ be a nonzero column vector in R^n, the n-dimensional Euclidean real vector space, and let T be an operator on R^n defined by

$$T(\mathbf{u}) \ = \ \text{proj}_{\mathbf{v}} \mathbf{u},$$

the operator of orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

(a) (4%) Show that T is a linear operator.

(b) (4%) Say $\mathbf{u} = [u_1\, u_2\, \cdots u_n]^T$ and $\mathbf{v} = [v_1\, v_2\, \cdots v_n]^T$; write down the standard matrix $[T]$ for the linear operator T.

(c) (2%) Is the standard matrix $[T]$ invertible? Explain your reasonings.
(Hint: Is T an one-to-one map?)

42. (10%) **Linear Transformation vs. Subspace** ($**$)

(a) (5%) Prove: If T is a linear operator on R^n, then the set

$$V \ = \ \{T(\mathbf{u}) \ : \ \mathbf{u} \in R^n\}$$

is a subspace of R^n.

(b) (5%) Is the converse of (a) true? That is, if V is a subspace of R^n, then does it imply that T is a linear operator? Prove the statement or disprove by providing an example.

43. (10%) **Subspaces** ($**$)
Let V and W be subspaces of R^3 spanned by 3 and 2 vectors respectively, as shown below:

$$V = span\left(\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\4\\12\end{bmatrix}, \begin{bmatrix}3\\1\\8\end{bmatrix}\right\}\right), \quad W = span\left(\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\13\end{bmatrix}\right\}\right).$$

Is W a subspace of V? Explain your answer, showing all intermediate steps.

44. (10%) **Spanning** ($**$)
Show that $\mathbf{v}_1 = (2, 12, 8)$, $\mathbf{v}_2 = (2, 4, 1)$, $\mathbf{v}_3 = (-2, 4, 10)$ and $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 16, 18)$ span the same subspace of R^3, i.e. show that

$$span(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = span(\{\mathbf{w}_1, \mathbf{w}_2\})$$

and that it is a subspace of R^3.

45. (10%) **Linear Space and Dimension.** ($**$)
Let N_1 and N_2 be finite-dimensional vector subspaces of a (not necessarily finite dimensional) vector space V such that $N_1 \cap N_2 = \{\mathbf{0}\}$. Prove that $N_1 + N_2 = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in N_1, \mathbf{v} \in N_2\}$ is a finite dimensional vector subspace of V and

$$\dim(N_1 + N_2) = \dim N_1 + \dim N_2$$

46. (10%) **Rank and Nullity.** ($**$)
Let A be a matrix given below:

$$A = \begin{bmatrix} 15 & 9 & 14 & 13 \\ 19 & 23 & 10 & 1 \\ -10 & -2 & -12 & -14 \end{bmatrix}$$

Find $rank(A)$, $nullity(A)$, $rank(A^T)$, $nullity(A^T)$, $det(\left(AA^T\right)^{2005})$.

47. (10%) **Inner Product Space.** ($**$)
If $S \neq \varnothing$ is a subset of an inner product space V, let $S^\perp = \{\mathbf{x} : \mathbf{x} \in V,\ \mathbf{x} \perp \mathbf{z} \text{ for any } \mathbf{z} \in S\}$. Let $S_1, S_2 \neq \varnothing$ be subsets of V, prove that $(S_1 \cup S_2)^\perp = S_1^\perp \cap S_2^\perp$.

48. (10%) **Eigenvalue, Eigenspace and Diagonalization.** ($*$)
Let

$$A = \begin{bmatrix} -7 & -9 & 3 \\ 2 & 4 & -2 \\ -3 & -3 & -1 \end{bmatrix}$$

(a) (0%) Find the eigenvalues of the matrix A.

(b) (0%) Find a basis for each eigenspace of the matrix A.

(c) (0%) Find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

49. (10%) **Change of Basis.** ($**$)
Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) (4%) Find the transition matrix $P_{B,B'}$ from B to B'.

(b) (4%) Find the transition matrix $P_{B',B}$ from B' to B.

(c) (2%) Compute the coordinate matrix $[\mathbf{w}]_{B'}$ where $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

50. (10%) **QR Decomposition.** ($**$)
Let A be a matrix given below:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}.$$

Perform the QR decomposition of A.

Differential Equations

51. (10%) **($*$)**
Solve the initial value problem

$$\frac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}, \quad y(0) = 0$$

52. (10%) **($*$)**
Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-\frac{x}{3}}, \quad y(0) = -1$$

53. (10%) **($*$)**
Find the general solution of the differential equation

$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$$

54. (10%) **($*$)**
Given that

$$y_1 = e^{\frac{x}{3}}$$

is a solution of the DE

$$6y'' + y' - y = 0$$

use reduction of order to find a second solution y_2.

55. (10%) **($*$)**
Solve the differential equation by *undetermined coefficients*

$$y'' + 2y' - 24y = 16 - (x+2)e^{4x}$$

56. (10%) **($**$)**
Solve the initial value problem

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0, \quad y(2) = 0$$

57. (10%) **($**$)**
Use Euler's method to obtain a two-decimal approximation of the indicated value. Let the incremental interval $h = 0.05$.

$$y' = 2x - y + 1, \quad y(1) = 5; \qquad y(1.2)$$

58. (10%) **($*$)**
Solve the IVP

$$y'' - 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 5$$

59. (10%) **($**$)**

(a) (3%) Verify that

$$y = -\frac{1}{x+c}$$

is a one-parameter family of solutions of the 1st-order DE

$$y' = y^2 \tag{1}$$

(b) (3%) Use the conditions in *Existence and Uniqueness Theorem* to show that the DE (1) has a unique solution with initial condition $y(0) = 1$.

(c) (4%) Find the solution of the DE (1) with the initial condition given in part (b), and determine the largest interval I of definition for the solution.

60. (10%) **($***$)**

(a) (5%) Sketch a direction field for

$$y' = 1 + 2xy$$

in the square $-2 \leq x \leq 0, 0 \leq y \leq 2$. (Sketch only for the integer grid, totally 25 points.)

(b) (5%) Using the direction field, sketch your guess for the solution curve passing through the point $(-2, 2)$.

61. (10%) **($***$)**

(a) (5%) Construct a direction field for the DE

$$y' = x - y$$

for the integer points at $x = -2, -1, 0, 1, 2$, $y = -2, -1, 0, 1, 2$, (totally 25 points).

(b) (5%) Use the result in part (a) to sketch an approximate solution curve that satisfies the initial condition $y(1) = 0$.

62. (10%) **($**$)**

Solve the DE

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

63. (10%) **($*$)**

Solve the following IVP using *variation of parameters*

$$2y'' + y' - y = x + 1, \quad y(0) = 1,\ y'(0) = 0$$

64. (10%) **($**$)**

Find the Frobenius series solutions of the differential equation:

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0$$

65. (10%) **($*$)**
Find $\mathscr{L}\{f(t)\}$ by definition if

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

66. (10%) **($*$)**
Find

$$\mathscr{L}\left\{t^2 \sin kt\right\}$$

67. (10%) **($**$)**
Use the *Laplace transform* to solve the integral equation

$$f(t) = \cos t + \int_0^t e^{-\tau} f(t-\tau) d\tau$$

68. (10%) **($**$)**
Use the improved Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$.

$$y' = y - y^2, \quad y(0) = 0.5; \quad y(0.5)$$

69. (10%) **($*$)**
Show that the functions

$$f_1(x) = e^x \quad \text{and} \quad f_2(x) = xe^{-x} - e^{-x}$$

are orthogonal on the interval $[0, 2]$.

70. (10%) **($**$)**
Expand the function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ x, & \text{if } 0 \le x < \pi \end{cases}$$

in a Fourier series

71. (10%) **($*$)**
Find a particular solution of the equation

$$y'' + y = \tan x$$

using *variation of parameters*.

72. (10%) **($**$)**
Use *power series method* to solve the differential equation:

$$y' + 2y = 0$$

73. (10%) **($*$)**
Use the *Laplace transform* to solve the initial-value problem

$$x'' + 4x = \sin 3t, \quad x(0) = 0,\ x'(0) = 0$$

74. (10%) **($**$)**
Find the convolution of

$$f(t) = \sin 2t \quad \text{and} \quad g(t) = e^t$$

using the *Laplace transform*.

75. (10%) **($**$)**
Evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

76. (10%) **($**$)**
Solve the initial-value problem

$$y'' + 2y' = \delta(t-1), \quad y(0) = 0,\ y'(0) = 1$$

77. (10%) **($***$)**
Use the method of Frobenius to obtain two linearly independent series solutions about $x = 0$ for the differential equation

$$9x^2y'' + 9x^2y' + 2y = 0$$

78. (10%) **($**$)**
Find the general solution in powers of x of

$$(x^2 - 4)y'' + 3xy' + y = 0$$

Then find the particular solution with $y(0) = 4$, $y'(0) = 1$.

79. (10%) **($**$)**
Solve

$$\mathbf{X}' = \begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix} \mathbf{X}$$

subject to

$$\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

80. (10%) **($*$)**
Use an *exponential matrix* to find the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} 4 & 3 \\ -4 & -4 \end{pmatrix} \mathbf{X}$$

81. (10%) **($**$)**
Solve

$$\mathbf{X}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-2t} \\ 1 \end{pmatrix}$$

using variation of parameters.

82. (10%) **($*$)**
Use the *Laplace transform* to solve the initial-value problem

$$y'' - 8y' + 20y = te^t, \quad y(0) = 0,\ y'(0) = 0$$

83. (10%) **($**$)**
Solve the initial-value problem

$$x^2y'' - 5xy' + 10y = 0, \quad y(1) = 1,\ y'(1) = 0$$

84. (10%) **($***$)**
Find the eigenvalues and and associated eigenfunctions of the Sturm-Liouville problem

$$\begin{array}{ll} y'' + \lambda y = 0, & (0 < x < L) \\ y'(0) = 0, & y(L) = 0 \end{array}$$

85. (10%) $***$**)**
Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

86. (10%) **($***$)**
Use separation of variables to solve the partial differential equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

87. (10%) **($***$)**
Use separation of variables to solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

88. (10%) **($*$)**
Classify the given partial differential equation as hyperbolic, parabolic, or elliptic.

(a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(b) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - 6\dfrac{\partial u}{\partial y} = 0$

89. (10%) **(∗∗)**
Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sin x, & \text{if } 0 \le x \le \pi \\ 0, & \text{if } x > \pi \end{cases}$$

90. (10%) **(∗ ∗ ∗)**
Solve the heat equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0$$

subject to

$$u(x,0) = f(x), \text{ where } f(x) = \begin{cases} A, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

91. (10%) **(∗)**
Find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

$$\frac{dP}{dt} + 2tP = P + 4t - 2$$

92. (10%) **(∗ ∗ ∗)**
A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h$$

where k and h are positive constants.

(a) (4%) Solve the DE subject to $P(0) = P_0$.

(b) (3%) Describe the behavior of the population $P(t)$ for increasing time in the three cases $P_0 > h/k, P_0 = h/k, 0 < P_0 < h/k$.

(c) (3%) Use the results from part (b) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct, then find T.

93. (10%) ∗∗**)**
Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation

$$x^2y'' - 4xy' + 6y = \ln x^2$$

94. (10%) **(∗)**
Solve the given initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos\gamma t, \quad x(0) = 0,\ x'(0) = 0$$

95. (10%) **($*$)**
Solve the given differential equation by using an appropriate substitution

$$\frac{dy}{dx} - y = e^x y^2$$

96. (10%) **($*$)**
Solve the given differential equation by undetermined coefficients

$$y^{(4)} - y'' = 4x + 2xe^{-x}$$

97. (10%) **($*$)**
Solve the differential equation by variation of parameters, subject to the initial conditions

$$y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}, \quad y(0) = 1,\ y'(0) = 0$$

98. (10%) **($*$)**
Solve the given differential equation by using an appropriate substitution

$$\frac{dy}{dx} = \sin(x + y)$$

99. (10%) **($*$)**
Solve the given initial-value problem

$$(e^x + y)dx + (2 + x + ye^y)dy = 0, \quad y(0) = 1$$

100. (10%) **($*$)**
Solve each differential equation by variation of parameters

$$y'' - 2y' + y = \frac{e^x}{1 + x^2}$$

Probability

101. (10%) **($*$)**
A random experiment consists of selecting two balls in succession from an urn containing two black balls and one white ball.
   (a) (3%) Specify the sample space for this experiment.
   (b) (3%) Suppose that the experiment is modified so that the ball is immediately put back into the urn after the first selection. What is the sample space now?
   (c) (4%) What is the relative frequency of the outcome (black, black) in a large number of repetitions of the experiment in part (a)? In part (b)?

102. (10%) **($*$)**
An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability of the second ball is white.

103. (10%) **($*$)**
An urn contains 100 balls and there are numbers 1,2,...,100 on these 100 balls. Suppose that every ball is selected equally. 20 balls are selected at random from the urn. Let X be the sum of the outcomes. Find the expected value of X.

104. (10%) **($*$)**
Let the events A and B have $P[A] = x$, $P[B] = y$, and $P[A \bigcap B] = z$. Use Venn diagrams to find $P[A^c \bigcup B^c]$, $P[A \bigcap B^c]$, $P[A^c \bigcap B]$ and $P[A^c \bigcap B^c]$.

105. (10%) **($*$)**
A binary transmission system sends a “0” bit using a -1 voltage signal and a “1” bit by transmitting a +1. The received signal is corrupted by noise that has a Laplacian distribution with parameter α. Assume that “0” bits and “1” bits are equiprobable.
   (a) (5%) Find the pdf of the received signal $Y = X + N$, where X is the transmitted signal, given that a “0” was transmitted; that a “1” was transmitted.
   (b) (5%) Suppose that the receiver decides a “0” was sent if $Y < 0$, and a “1” was sent if $Y \geq 0$. What is the overall probability of error?

106. (10%) **($*$)**
Let be uniformly distributed in the interval $(0, 2\pi)$. Let $X = \cos\Theta$ and $Y = \sin\Theta$. Show that X and Y are uncorrelated.

107. (10%) **($**$)**
Let the random variable Y be defined by $Y = X^2$, where $X \sim N(0, 1)$. Find the cdf and pdf of Y .

108. (10%) **($*$)**
Prove the memoryless property of exponential distribution.

109. (10%) **($*$)**
Suppose that orders at a restaurant are i.i.d. random variables with mean μ=\$8 and standard deviation σ=\$2. Estimate the probability that the first 100 customers spend a total of more than \$840. Estimate the probability that the first 100 customers spend a total of between \$780 and \$820.

110. (10%) **($*$)**
Let X and Y be independent random variables each geometrically distributed with parameter p .

(a) (2.5%) Find the distribution of min(X, Y) .

(b) (2.5%) Find P(min$(X, Y) = X$) = $P(Y \geq X)$

(c) (2.5%) Find the distribution of $X + Y$

(d) (2.5%) Find $P(Y = y|X + Y = z)$ for $y = 0, 1, \ldots, z$

111. (10%) **($*$)**
We have two coins, one that is fair and thus lands heads up with probability $1/2$ and one that is unfair and lands heads up with probability p , $p > 1/2$. One of the coins is selected randomly and tossed n times yielding n straight tails. What is the probability that the unfair coin was selected?

112. (10%) **($**$)**
Find the probability that the sum of the outcomes of five tosses of a die is 20.

113. (10%) **($**$)**
Given the following joint pdf

$$f_{X,Y}(x, y) = \begin{cases} ce^{-x}e^{-y}, & 0 \leq x \leq 2y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

(a) (5%) Find the normalization constant c, $P[X + Y \leq 1]$,$f_Y(y|x)$,$E[X|y]$,$E[XY]$,$COV(X, Y)$, and $\rho_{X,Y}$.

(b) (5%) Let V and W be obtained from (X, Y) by

$$\begin{cases} V = X + 2Y \\ W = X - Y \end{cases}$$

Find the joint pdf of V and W .

114. (10%) **($**$)**
Let $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \cdots, X_n$ are i.i.d. random variables with finite mean m and finite variance σ^2. Let $Z_n = \frac{S_n - nm}{\sigma\sqrt{n}}$. Prove that $\lim_{n\longrightarrow\infty} Z_n \sim N(0, 1)$.

115. (10%) **($**$)**
Let U and V are independent zero-mean, unit-variance Gaussian random variables, and let $X = U + V$, $Y = 2U + V$. Find $E[XY]$.

116. (10%) **($**$)**
Let $X = (X_1, X_2)$ be the jointly Gaussian random variables with mean vector and covariance matrix given by

$$m_X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad K_X = \begin{bmatrix} \frac{9}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{6}{5} \end{bmatrix}$$

(a) (2.5%) Find the pdf of X in matrix notation.

(b) (2.5%) Find the marginal pdfs of X_1 and X_2.

(c) (2.5%) Find a transformation A such that the vector $Y = AX$ consists of independent Gaussian random variables.

(d) (2.5%) Find the joint pdf of Y.

117. (10%) **($**$)**
Two players having respective initial capital \$5 and \$10 agree to make a series of \$1 bets unit one of them goes broke. Assume the outcomes of the bets are independent and both players have probability 1/2 winning any given bet. Find the probability that the player with the initial capital of \$10 goes broke. Find the expected number of bets.

118. (10%) **($**$)**
At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men that select their own hats.

119. (10%) **($**$)**
Let N, for $N \geq 1$ and such that $E[N]$ exists, be a random variable that is a stopping time for the sequence of independent and identically distributed random variables X_i, $i = 1, 2, \cdots$, having finite expectation. Let

$$S_N = X_1 + X_2 + \cdots + X_N$$

Then show that

$$E[S_N] = E[X]\,E[N]$$

120. (10%) **($**$)**
The number of customers that arrive at a service station during a time t is a Poisson random variable with parameter βt. The time required to service each customer is an exponential random variable with parameter α. Find the pmf for the number of customers N that arrive during the service time T of a specific customer. Assume that the customer arrivals are independent of the customer service time.

121. (10%) **($**$)**
Find the pdf of the sum $Z = X + Y$ of two zero-mean, unit-variance Gaussian random variables with correlation coefficient $\rho = -1/2$.

122. (10%) **($**$)** (Markov's Inequality)
Prove that if X is a random variable that takes only nonnegative values, then for any value $a > 0$

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

123. (10%) **($*\,*\,*$)**
Let X_1, X_2, X_3 are independent, exponentially distributed random variables with parameter λ_1, λ_2, and λ_3 respectively. Find the pdf of $X_1 + X_2 + X_3$.

124. (10%) **($*\,*\,*$)**
Let $Z = X/Y$. Find the pdf of Z if X and Y are independent and both exponentially distributed with mean one.

125. (10%) **($*\,*\,*$)**
Let $X_1, X_2, \cdots, X_n$ be independent exponential random variables with pdf $f_{X_i}(x) = \lambda_i e^{-\lambda_i x}, x \geq 0, \lambda_i \geq 0$. Define

$$Y_n = \min\{X_1, X_2, \cdots, X_n\}, Z_n = \max\{X_1, X_2, \cdots, X_n\}$$

(a) (5%) Find the cdf of Y_n and Z_n.

(b) (5%) Find the probability that X_i is the smallest one among $X_1, X_2, \cdots, X_n$.

126. (10%) **($*\,*\,*$)**
Let $X_1, X_2, \cdots$ be independent exponential random variables with mean $1/\lambda$ and let N be a discrete random variable with $P(N = k) = (1-p)\,p^{k-1}, k = 1, 2, \cdots$, where $0 < p < 1$ (i.e. N is a shifted geometric random variable). Show that S defined as

$$S = \sum_{n=1}^{N} X_n$$

is again exponentially distributed with parameter $(1-p)\,\lambda$.

127. (10%) **($*\,*\,*$)**
The points A, B, C are independent and uniformly distributed on a circle with its center at O. Find the probability that the center O lies in the interior of triangle $\triangle ABC$?

128. (10%) **($*\,*\,*$)**
A coin with the probability of head $p\{H\} = p = 1 - q$ is tossed n times.

(a) (5%) Find the probability that k heads are observed up to the n-th tossing but not earlier.

(b) (5%) Show that the probability that the number of heads is even equals $0.5\,[1 + (q-p)^n]$.

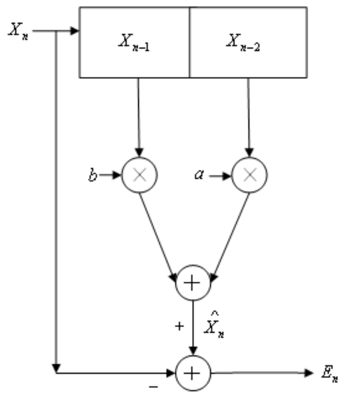
129. (10%) **($*\,*\,*$)**
A point (X, Y) is selected randomly from the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$

(a) (5%) Find the joint probability density function of X and Y.

(b) (5%) Evaluate $E(X|Y = y)$, $VAR(X|Y = y)$.

130. (10%) **($*\,*\,*$)**
Let X be a zero-mean, unit-variance Gaussian random variable an let Y be a chi-square random variable with n degrees of freedom. Assume that X and Y are independent. Find the pdf of $V = X/\sqrt{Y/n}$.

131. (10%) **($*\,*\,*$)**
Let $X_1, X_2, \cdots$, be a sequence of samples of a speech voltage waveform, and suppose that the samples are fed into the second-order predictor shown in the following figure. Find the set of predictors a and b that minimize the mean square value of the predictor error.



132. (10%) **($*\,*\,*$)**
Let $X = (X_1, X_2, \cdots, X_n)$ be zero-mean jointly Gaussian random variables. Show that

$$E[X_1X_2X_3X_4] = E[X_1X_2]\,E[X_3X_4] + E[X_1X_3]\,E[X_2X_4] + E[X_1X_4]\,E[X_2X_3]$$

133. (10%) **($*$)**
Consider a function

$$f(x) = \frac{1}{\sqrt{\pi}} e^{\left(-x^2+x-a\right)}, \quad -\infty < x < \infty$$

Find the value of a such that $f(x)$ is a pdf of a continuous r.v. X.

134. (10%) **($**$)**
Find the mean and variance of a Rayleigh r.v. defined by

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x > 0 \\ 0 & x < 0 \end{cases}$$

135. (10%) **($**$)**
Let $X = N(0; \sigma^2)$. Find $E[X|X > 0]$ and $Var(X|X > 0)$.

136. (10%) **($**$)**
Suppose we select one point at random from within the circle with radius R. If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen (Fig. 3-8), then (X, Y) is a uniform bivariate r.v. with joint pdf given by

$$f_{XY}(x, y) = \begin{cases} k, & x^2 + y^2 \le R^2 \\ 0, & x^2 + y^2 > R^2 \end{cases}$$

where k is a constant.

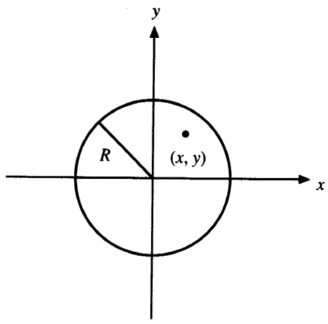


Fig. 3-8

(a) (3%) Determine the value of k.

(b) (3%) Find the marginal pdfs of X and Y.

(c) (4%) Find the probability that the distance from the origin of the point selected is not greater than a.

137. (10%) **(∗)**
Suppose the joint pmf of a bivariate r.v. (X, Y) is given by

$$P_{XY}(x_i, y_j) = \begin{cases} \frac{1}{3}, & (0,1), (1,0), (2,1) \\ 0, & otherwise \end{cases}$$

(a) (5%) Are X and Y independent?

(b) (5%) Are X and Y uncorrelated?

138. (10%) **(∗)**
Let X and Y be two r.v.'s with joint pdf $f_{XY}(x, y)$ and joint cdf $F_{XY}(x, y)$. Let $Z = \max(X, Y)$.

(a) (5%) Find the cdf of Z?

(b) (5%) Find the pdf of Z if X and Y are independent.

139. (10%) **(∗∗)**
Let X and Y be two r.v.s with joint pdf $f_{XY}(x, y)$. Let

$$R = \sqrt{X^2 + Y^2} \quad \Theta = \tan^{-1} \frac{Y}{X}$$

Find $f_{R\Theta}(r, \theta)$ in terms of $f_{XY}(x, y)$.

140. (10%) **(∗)**
Let X and Y be two r.v.'s with joint pdf $f_{XY}(x, y)$. Let

$$\begin{aligned} Y_1 &= X_1 + X_2 + X_3 \\ Y_2 &= X_1 - X_2 \\ Y_3 &= X_2 - X_3 \end{aligned}$$

Determine the joint pdf of Y_1, Y_2, and Y_3.

141. (10%) **(∗∗)**
Show that if $X_1, \cdots, X_n$ are independent Poisson r.v.'s X_i having parameter λ_i, then $Y = X_1 + \cdots + X_n$ is also a Poisson r.v. with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.

142. (10%) **($*$)**
Let X be a uniform r.v. over $(-1, 1)$. Let $Y = X^n$.

(a) (5%) Calculate the covariance of X and Y.

(b) (5%) Calculate the correlation coefficient of X and Y.

143. (10%) **($*$)**
A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities .001, .005, and .01, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A; that the manufacturer was C.

144. (10%) **($*$)**
A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{4/10, 3/10, 2/10, 1/10\}$.

(a) (3%) Find the pmf of the output Y of the channel.

(b) (3%) What is the probability that the output of the channel is equal to the input of the channel?

(c) (4%) What is the probability that the output of the channel is positive?

145. (10%) **($*$)**
Let X be the number of successes in n Bernoulli trials where the probability of success is p. Let $Y = X/n$ be the average number of successes per trial. Apply the Chebyshev's inequality to the event $\{|Y - p| > a\}$. What happens as $n \to \infty$?

146. (10%) **($*$)**
Let X and Y denote the amplitude of noise signals at two antennas. The random vector (X, Y) has the joint pdf

$$f(x, y) = axe^{-ax^2/2}bye^{-by^2/2}, \quad x > 0, y > 0, a > 0, b > 0$$

(a) (3%) Find the joint cdf.

(b) (3%) Find $P[X > y]$.

(c) (4%) Find the marginal pdf's.

147. (10%) **($*$)**
Let X and Y have joint pdf

$$f_{XY}(x, y) = k(x + y), \quad \text{for } 0 \le x \le 1, 0 \le y \le 1.$$

(a) (2.5%) Find k.

(b) (2.5%) Find the joint cdf of (X, Y).

(c) (2.5%) Find the marginal pdf of X and Y.

(d) (2.5%) Find $P[X < Y]$, $P[Y < X^2]$, $P[X + Y > 0.5]$.

148. (10%) **($*$)**
Find $E\left[X^2e^Y\right]$ where X and Y are independent random variables, X is a zero-mean, unit-variance Gaussian random variable, and Y is a uniform random variable in the interval $[0,3]$.

149. (10%) **($*$)**
Signals X and Y are independent. X is exponentially distributed with mean 1 and Y is exponentially distributed with mean 1.

(a) (5%) Find the cdf of $Z = |X - Y|$.

(b) (5%) Use the result of part (a) to find $E[Z]$.

150. (10%) **($**$)**
The random variables X and Y have the joint pdf

$$f_{XY}(x,y) = e^{-(x+y)}, \quad \text{for } 0 < y < x < 1.$$

Find the pdf of $Z = X + Y$.